# REAL OPTION VALUE

## CHAPTER 14 INCENTIVE OPTIONS

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Incentive options can be viewed using the toolkit implicit in previous chapters of real payoff diagrams, entry and exit options, and perpetual American puts and calls. Incentive options may be granted (or required by) governments to encourage early investment in "desirable" projects such as renewable energy facilities, infrastructure investments like roads, bridges and other transportation, and in general public-private partnerships governing new facilities like schools, hospitals, and recreation areas.

These incentive options are classified as (i) proportional revenue (or price and/or quantity) subsidies, where the market price and/or the quantity of production is uncertain or low, but the subsidy is proportional to the quantity produced (ii) supplementary revenue (or price and/or quantity) subsidies, where the market price and/or the quantity of production and/or the exogenous subsidy is uncertain (iii) revenue floors and ceilings, where the subsidy is related over time to the actual quantities produced or market prices. Examples of (i) are so-called Feed-in-Tariffs "FiT" which are fixed amount subsidies per unit production, (ii) renewable "green" certificates, which have an uncertain value but are usually allocated per unit of production, and (iii) government minimum revenue guarantees, sometimes accompanied by maximum revenue ceilings.

In addition, governments provide incentives for free or at low cost (sport stadiums, concessions, priority access, protection through tariffs, quotas or security) in order to encourage "desirable" activities, or investment cost reliefs, consisting of direct grants and soft loans, tax credits or excess depreciation, which are not directly considered here, except in examining sensitivities of thresholds and real option value to changes in investment costs or taxation. Some of these incentives can be evaluated in terms of the real option value compared to that paid to the government (taxes, concession and user fees and royalties) weighted against the immediate or eventual cost for the government. Also it is interesting to study the effect on the real option value, and on the threshold that justifies immediate investment, of price, quantity and subsidy changes. Who gets/gives what, when, how, and why are almost always critical considerations in incentive options.

### 14.1 Proportional Subsidies

This section considers a menu of possible characteristic subsidies, first where there is no subsidy (Model 1); then assuming there is a permanent subsidy proportional to the revenue (Model 2); then assuming there is a retractable subsidy proportional to the revenue (Model 3A), and finally assuming there is only the possibility of a permanent subsidy (Model 3B), as suggested in the Adkins and Paxson (2015), Appendix.

**Proportional Stochastic Revenue Models**

Consider a perpetual opportunity to construct an electricity generating facility producing Q MWhrs/pa, using solar power, at a fixed investment cost $K$. This investment cost is treated as irreversible or irrecoverable once incurred. The real option value of this investment opportunity, denoted by ROV, depends on the amount of output Q, and the price per unit of output, denoted by $P$, P*Q=R revenue, assuming no operating or maintenance costs or taxes. R is assumed to be stochastic and to follow a geometric Brownian motion process:

$$\mathrm{dR} = \theta_R \mathrm{Rd}t + \sigma_R R\mathrm{d}Z \tag{1}$$

where $\theta_R$ denotes the instantaneous risk neutral drift parameter (equals δ the asset yield), $\sigma_R$ the instantaneous volatility, and $\mathrm{d}Z$ the standard Wiener process. The differential equation representing the value to invest for an inactive investor with an appropriate investment opportunity (based perhaps on approval for the facility or a concession for infrastructure) is:

$$\frac{1}{2}\sigma_R^2 R^2 \frac{\partial^2 ROV_1}{\partial R^2} + \theta_R R \frac{\partial ROV_1}{\partial R} - rROV_1 = 0. \tag{2}$$

where $r$ is the risk-free rate. Adkins and Paxson (2015) show that the solution to (2) is:

$$ROV_1 = B_1 R^{\beta_1}. \tag{3}$$

$\beta_1$ is the power parameter for this option value function. Since there is an incentive to invest when R is sufficiently high but a disincentive when sufficiently low, the power parameter value is positive. Also, the power parameter is determined using the characteristic root equation (which is the positive root of a simple quadratic equation) found by substituting (3) in (2):

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{(\frac{r-\delta}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} . \tag{4}$$

After the investment, the solar plant generates revenue equaling (1+τ)*R, (so S=τR) where τ is the permanent subsidy proportional to the revenue sold (τ=0 indicates no possible subsidy). So from (2), the valuation relationship for the operational state is:

$$\frac{1}{2}\sigma_R^2 R^2 \frac{\partial^2 ROV_1}{\partial R^2} + \theta_R R \frac{\partial ROV_1}{\partial R} + (1+\tau)R - rROV_1 = 0. \tag{5}$$

After the investment (K), the solution to (5) is:

$$\frac{(1+\tau)R}{r-\theta_R} . \tag{6}$$

**Model 1**

The subsidy is set to equal zero in Model 1. If the threshold revenue signaling an optimal investment is denoted by $\hat{R}_1$, then:

$$\hat{R}_1 = \frac{\beta_1}{\beta_1 - 1} K\left(r - \theta_R\right). \tag{7}$$

The value for the investment opportunity is defined by:

$$ROV_1 = \begin{cases} B_1 R^{\beta_1} & \text{for } R < \hat{R}_1, \\ \dfrac{R}{r-\theta_R} - K & \text{for } R \geq \hat{R}_1. \end{cases} \tag{8}$$

where:
$$B_1 = \frac{\hat{R}_1^{1-\beta_1}}{\beta_1 (r-\theta_R)}. \tag{9}$$

**Model 2**

For a positive proportional permanent subsidy $\tau$ , the corresponding results are:

$$\hat{R}_2 = \frac{\beta_1}{\beta_1 - 1} K \frac{(r-\theta_R)}{(1+\tau)}, \tag{10}$$

$$ROV_2 = \begin{cases} B_2 R^{\beta_1} & \text{for } R < \hat{R}_2, \\ \dfrac{R(1+\tau)}{r-\theta_R} - K & \text{for } R \geq \hat{R}_2, \end{cases} \tag{11}$$

$$B_2 = \frac{(1+\tau)\hat{R}_2^{1-\beta_1}}{\beta_1 (r-\theta_R)} \tag{12}$$

**Model 3A**

The probability of a sudden unexpected withdrawal of the subsidy is denoted by $\lambda$. If the revenue threshold signaling an optimal investment is denoted by $\hat{R}_3$, then its solution is found implicitly from:

$$\hat{R}_3 = \frac{\beta_3}{\beta_3 - 1} K \frac{r-\theta_R}{1+(1-\lambda)\tau} + B_1 \hat{R}_3^{\beta_1} \frac{\beta_3 - \beta_1}{\beta_3 - 1} \tag{13}$$

where $B_1$ is from (9). The value for the investment opportunity is specified by:

$$ROV_3 = \begin{cases} B_3 R^{\beta_3} + B_1 R^{\beta_1} \quad \text{for} \quad R < \hat{R}_3, \\ \dfrac{R\left(1+\left(1-\lambda\right)\tau\ \right)}{r-\theta_R} - K \quad \text{for} \quad R \geq \hat{R}_3, \end{cases} \tag{14}$$

where:

$$B_3 = \frac{(1+\left(1-\lambda\right)\tau_M)\hat{R}_3^{1-\beta_3}}{\beta_3(r-\theta_R)} - \frac{\beta_1}{\beta_3} B_1 \hat{R}_3^{\beta_1-\beta_3}. \tag{15}$$

$\beta_3$ is the positive root of (4) with $\lambda$ added to r. For $\lambda = 0$, when there is no likelihood of the subsidy being withdrawn unexpectedly, $\beta_3 = \beta_1$ and Model 3A simplifies to the Model 2 solution.

## Model 3B

The probability of a sudden unexpected introduction of a permanent subsidy is denoted by $\lambda$. If the revenue threshold signaling an optimal investment is denoted by $\hat{R}_4$, then its solution is found implicitly from:

$$\hat{R}_3 = \frac{\beta_3}{\beta_3 - 1} \frac{r-\theta_R}{1+\lambda\tau} (K + \frac{\lambda}{r+\lambda} B_2 \hat{R}_2^{\beta_1}) \tag{16}$$

where $B_2$ is from (12). The value for the investment opportunity is specified by:

$$ROV_4 = \begin{cases} B_4 R^{\beta_3} + \dfrac{\lambda}{r+\lambda} B_2 R^{\beta_1} \quad \text{for} \quad R < \hat{R}_4, \\ \dfrac{R\left(1+\lambda\tau\ \right)}{r-\theta_R} - K \quad \text{for} \quad R \geq \hat{R}_4, \end{cases} \tag{17}$$

where:

$$B_3 = \frac{(1+\lambda\tau\ )\hat{R}_4^{1-\beta_3}}{\beta_3(r-\theta_R)}. \tag{18}$$

For $\lambda = 0$, when there is no likelihood of an unexpected introduction of a permanent proportional subsidy, Model 3B simplifies to the Model 0 solution. It is easy to put these formulae into Excel as shown in Figures 1, 2, 3, 4A and 4B below.

Figure 1

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **REVENUE MODEL 1** | | | |
| 2 | INPUT | Stochastic R | | EQ |
| 3 | P | 22.50 | Per MWhr | |
| 4 | Q | 10.00 | MWhrs/per annum | |
| 5 | R | 225.00 | B3*B4 | |
| 6 | K | 4000.00 | Per Capacity of 10 MWhrs/per annum | |
| 7 | $\sigma$ | 0.20 | Template | |
| 8 | r | 0.08 | Given | |
| 9 | $\theta$ | 0.04 | Template | |
| 10 | $\tau$ | 0.00 | NO SUBSIDY | |
| 11 | $r-\theta$ | 0.04 | B8-B9 | |
| 12 | $\lambda$ | 0.00 | Probability | |
| 13 | OUTPUT | | | |
| 14 | $ROV_1$ | 2456.34 | IF(B5<B18,B17*(B5^B16),B15) | 8 |
| 15 | V-K | 1625.00 | ((1+B10)*B5/B11)-B6 | |
| 16 | $\beta_1$ | 1.5616 | | 4 |
| 17 | $B_1$ | 0.5215 | (B18^(1-B16))/(B16*B11) | 9 |
| 18 | $R_1$* | 444.92 | B6*B11*(B16/(B16-1)) | 7 |
| 19 | $\beta_1$ | (1/B7^2)*(-(B9-0.5*(B7^2))+SQRT((B9-0.5*(B7^2))^2+(2*B8)*(B7^2))) | | |
| 20 | ODE | 0.0000 | 0.5*(B7^2)*(B5^2)*B22+B9*B5*B21-B8*B14 | 2 |
| 21 | $\Delta$ROV | 17.0476 | B16*B17*(B5^(B16-1)) | |
| 22 | $\Gamma$ROV | 0.0425 | B16*(B16-1)*B17*(B5^(B16-2)) | |

Figure 2 illustrates a subsidy of $\tau$=1, which results in a threshold R*=R, justifying immediate investment.

Figure 2

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **REVENUE MODEL 2** | | | |
| 2 | INPUT | Stochastic R | | EQ |
| 3 | P | 22.50 | | |
| 4 | Q | 10.00 | | |
| 5 | R | 225.00 | B3*B4 | |
| 6 | K | 4000.00 | | |
| 7 | $\sigma$ | 0.20 | | |
| 8 | r | 0.08 | | |
| 9 | $\theta$ | 0.04 | | |
| 10 | $\tau$ | 1.00 | SUBSIDY | |
| 11 | $r-\theta$ | 0.04 | B8-B9 | |
| 12 | $\lambda$ | 0.00 | Probability of withdrawal | |
| 13 | OUTPUT | | | |
| 14 | $ROV_2$ | 7250.00 | IF(B5<B18,B17*(B5^B16),B15) | 11 |
| 15 | V-K | 7250.00 | ((1+B10)*B5/B11)-B6 | |
| 16 | $\beta_1$ | 1.5616 | | |
| 17 | $B_2$ | 1.5392 | ((1+B10)*B18^(1-B16))/(B16*B11) | 12 |
| 18 | $R_2$* | **222.46** | (B6*B11/(1+B10))*(B16/(B16-1)) | 10 |
| 19 | $\beta_1$ | (1/B7^2)*(-(B9-0.5*(B7^2))+SQRT((B9-0.5*(B7^2))^2+(2*B8)*(B7^2))) | | |

Figure 3 shows that when the probability of subsidy withdrawal is zero, Model 3 is reduced to Model 2 in Figure 2.

Figure 4A shows Model 3 with a positive probability of withdrawal, which reduces R* significantly, a "flighty bird in hand" motivates early investment.

Figure 3

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **REVENUE MODEL 3A** | EQ |
| 2 | INPUT | Stochastic R | | |
| 3 | P | 22.50 | | |
| 4 | Q | 10.00 | | |
| 5 | R | 225.00 | B3*B4 | |
| 6 | K | 4000.00 | | |
| 7 | $\sigma$ | 0.20 | | |
| 8 | r | 0.08 | | |
| 9 | $\theta$ | 0.04 | | |
| 10 | $\tau$ | 1.00 | | |
| 11 | $r-\theta$ | 0.04 | B8-B9 | |
| 12 | $\lambda$ | 0.00 | Probability | |
| 13 | OUTPUT | | | |
| 14 | $ROV_3$ | 7250.00 | IF(B5<B18,B17*(B5^B16)+B24*(B5^B23),B15) | 14 |
| 15 | V-K | 7250.00 | ((1+(1-B12)*B10)*B5/B11)-B6 | |
| 16 | $\beta_3$ | 1.5616 | | 4 |
| 17 | $B_3$ | 1.0178 | | 15 |
| 18 | $R_3$* | **222.46** | | |
| 19 | Solver | 0.0000 | Set B19=0, Changing B18 | 13 |
| 20 | $\beta_1$ | 1.5616 | | |
| 21 | $B_1$ | 0.5215 | | |
| 22 | $R_1$* | 444.92 | | |
| 23 | $\beta_3$ | (1/B7^2)*(-(B9+B12-0.5*(B7^2))+SQRT((B9+B12-0.5*(B7^2))^2+(2*(B8+B12)*(B7^2)))) | | |
| 24 | $R_3$* | ((B6*B11)/(1+(1-B12)*B10))*(B16/(B16-1))+B24*(B18^B23)*((B16-B23)/(B16-1))-B18 | | |
| 25 | $B_3$ | ((1+(1-B12)*B10)*B18^(1-B16))/(B16*B11)-(B23/B16)*B24*(B18^(B23-B16)) | | |

Figure 4A

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **REVENUE MODEL 3A** | EQ |
| 2 | INPUT | Stochastic R | | |
| 3 | P | 22.50 | | |
| 4 | Q | 10.00 | | |
| 5 | R | 225.00 | B3*B4 | |
| 6 | K | 4000.00 | | |
| 7 | $\sigma$ | 0.20 | | |
| 8 | r | 0.08 | | |
| 9 | $\theta$ | 0.04 | | |
| 10 | $\tau$ | 1.00 | | |
| 11 | $r-\theta$ | 0.04 | B8-B9 | |
| 12 | $\lambda$ | 0.10 | Probability | |
| 13 | OUTPUT | | | |
| 14 | $ROV_3$ | 6687.50 | IF(B5<B18,B17*(B5^B16)+B24*(B5^B23),B15) | 14 |
| 15 | V-K | 6687.50 | ((1+(1-B12)*B10)*B5/B11)-B6 | |
| 16 | $\beta_3$ | 1.2426 | | 4 |
| 17 | $B_3$ | 11.9792 | | 15 |
| 18 | $R_3$* | **56.65** | | |
| 19 | Solver | 0.0000 | Set B19=0, Changing B18 | 13 |
| 20 | $\beta_1$ | 1.5616 | | |
| 21 | $B_1$ | 0.5215 | | |
| 22 | $R_1$* | 444.92 | | |
| 23 | $\beta_3$ | (1/B7^2)*(-(B9+B12-0.5*(B7^2))+SQRT((B9+B12-0.5*(B7^2))^2+(2*(B8+B12)*(B7^2)))) | | |
| 24 | $R_3$* | ((B6*B11)/(1+(1-B12)*B10))*(B16/(B16-1))+B24*(B18^B23)*((B16-B23)/(B16-1))-B18 | | |
| 25 | $B_3$ | ((1+(1-B12)*B10)*B18^(1-B16))/(B16*B11)-(B23/B16)*B24*(B18^(B23-B16)) | | |

Figure 4B

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **REVENUE MODEL 3B** | EQ |
| 2 | INPUT | Stochastic R | | |
| 3 | P | 22.50 | | |
| 4 | Q | 10.00 | | |
| 5 | R | 225.00 | B3*B4 | |
| 6 | K | 4000.00 | | |
| 7 | $\sigma$ | 0.20 | | |
| 8 | r | 0.08 | | |
| 9 | $\theta$ | 0.04 | | |
| 10 | $\tau$ | 1.00 | | |
| 11 | $r-\theta$ | 0.04 | B8-B9 | |
| 12 | $\lambda$ | 0.10 | Probability | |
| 13 | OUTPUT | | | |
| 14 | $ROV_4$ | 7108.92 | IF(B5<B18,B17*(B5^B16)+(B12/(B8+B12))*B22*(B23^B21),B15) | 17 |
| 15 | V-K | 2187.50 | (1+B12*B10)*B5/B11-B6 | |
| 16 | $\beta_3$ | 1.2426 | | 18 |
| 17 | $B_4$ | 3.7637 | ((1+B10*B12)*B18^(1-B16))/(B16*B11) | |
| 18 | $R_4$* | 1481.88 | | 16 |
| 19 | $\beta_3$ | (1/B7^2)*(-(B11+B12-0.5*(B7^2))+SQRT((B11+B12-0.5*(B7^2))^2+(2*(B8+B12)*(B7^2)))) | | |
| 20 | $R_4$* | (B16/(B16-1))*(B11/(1+B10*B12))*(B6+(B12/(B8+B12))*B22*(B23^B21)) | | |
| 21 | $\beta_1$ | 1.5616 | | |
| 22 | $B_2$ | 1.5392 | | |
| 23 | $R_2$* | 222.4621 | | |

## 14.2 Exogenous Subsidies

### Model 4 Stochastic Price, Subsidy and Quantity

Now consider a perpetual opportunity to construct a facility at a fixed investment cost $K$, where the subsidy is exogenous like a "green certificate". The value of this investment opportunity, denoted by $F_1$, depends on the amount of output sold per unit of time, denoted by Q, the market price per unit of output, denoted by $P$, and the subsidy per output unit, S. In the general model, all of these variables are assumed to be stochastic and are assumed to follow geometric Brownian motion processes (gBm):

$$\mathrm{d}X = \theta_X X\mathrm{d}t + \sigma_X X\mathrm{d}Z \qquad (1)$$

for $X \in \{P, S, Q\}$, where $\theta$ denotes the risk neutral instantaneous drift parameter, $\sigma$ the instantaneous volatility, and $\mathrm{d}Z$ the standard Wiener process. Potential correlation between the variables is represented by $\rho$.

The partial differential equation (PDE) representing the value to invest for an inactive firm with an appropriate perpetual investment opportunity (based on perhaps approval for the facility or a concession for infrastructure) is:

$$\begin{aligned}
&\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F_1}{\partial P^2} + \frac{1}{2}\sigma_Q^2 Q^2 \frac{\partial^2 F_1}{\partial Q^2} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 F_1}{\partial S^2} \\
&+PQ\rho_{PQ}\sigma_P\sigma_Q \frac{\partial^2 F_1}{\partial P \partial Q} + PS\rho_{PS}\sigma_P\sigma_S \frac{\partial^2 F_1}{\partial P \partial S} + QS\rho_{QS}\sigma_Q\sigma_S \frac{\partial^2 F_1}{\partial Q \partial S} \\
&+\theta_P P \frac{\partial F_1}{\partial P} + \theta_Q Q \frac{\partial F_1}{\partial Q} + \theta_S S \frac{\partial F_1}{\partial S} - rF_1 = 0.
\end{aligned} \tag{2}$$

where $r$ is the risk-free rate. Following Adkins and Paxson (2016), when P,Q, or S are below $\hat{P}, \hat{Q}, \hat{S}$ that justify immediate investment, the solution to (2) is:

$$ROV_1 = F_1 = A_1 P^{\beta_1} Q^{\gamma_1} S^{\eta_1}. \tag{3}$$

where $\beta_1$ , $\gamma_1$ and $\eta_1$ are the power parameters for this option value function. Since there is an incentive to invest when $P$ , Q and S are sufficiently high but a disincentive when these are sufficiently low, we expect that all power parameter values are positive. Also, the parameters are linked through the characteristic root equation found by substituting (3) in (2):

$$\begin{aligned}
&Q(\beta_1, \gamma_1, \eta_1) = \tfrac{1}{2}\sigma_P^2\beta_1(\beta_1 - 1) + \tfrac{1}{2}\sigma_Q^2\gamma_1(\gamma_1 - 1) + \tfrac{1}{2}\sigma_S^2\eta_1(\eta_1 - 1) + \\
&\rho_{PQ}\sigma_P\sigma_Q\beta_1\gamma_1 + \rho_{PS}\sigma_P\sigma_S\beta_1\eta_1 + \rho_{QS}\sigma_Q\sigma_S\gamma_1\eta_1 \\
&+\theta_P\beta_1 + \theta_Q\gamma_1 + \theta_S\eta_1 - r = 0
\end{aligned} \quad . \tag{4}$$

After the investment, the plant generates revenue equaling $PQ + SQ$, with the present value factor of parts of this net revenue denoted $k_{P,}$ $k_Q$ and $k_S$ (no operating costs or taxes) (life assumed to be T=20 years in the base case)[1].

$$k_P = \frac{1 - e^{-(r-\theta_P)*T}}{(r - \theta_P)}, k_{PQ} = \frac{1 - e^{-(r-\theta_P-\theta_Q)*T}}{(r - \theta_P - \theta_Q)} \tag{5}$$

$$k_Q = \frac{1 - e^{-(r-\theta_Q)*T}}{(r - \theta_Q)} \tag{6}$$

[1] This is the methodology in Boomsma and Linnerud (2015).

$$k_S = \frac{1-e^{-(r-\theta_S)*T}}{(r-\theta_S)}, k_{SQ} = \frac{1-e^{-(r-\theta_S-\theta_Q)*T}}{(r-\theta_S-\theta_Q)}, \tag{7}$$

The value matching relationship, when the real option value upon exercise is equal to the net present value of the investment (NPV), is:

$$A_1 \hat{P}^{\beta_1} \hat{Q}^{\gamma_1} \hat{S}_1^{\eta_1} = k_{PQ} \hat{P} \hat{Q} + k_{SQ} \hat{S}_1 \hat{Q} - K \tag{8}$$

The three associated smooth pasting conditions can be expressed as:

$$\beta_1 A_1 \hat{P}^{\beta_1} \hat{Q}^{\gamma_1} \hat{S}_1^{\eta_1} = k_{PQ} \hat{P} \hat{Q} \tag{9}$$

$$\gamma_1 A_1 \hat{P}^{\beta_1} \hat{Q}^{\gamma_1} \hat{S}_1^{\eta_1} = k_{PQ} \hat{P} \hat{Q} + k_{SQ} \hat{S}_1 \hat{Q} \tag{10}$$

$$\eta_1 A_1 \hat{P}^{\beta_1} \hat{Q}^{\gamma_1} \hat{S}_1^{\eta_1} = k_{SQ} \hat{S}_1 \hat{Q} \tag{11}$$

A quasi-analytical solution to the set of five equations 4-8-9-10-11 for 7 unknowns

$\hat{P}, \hat{Q}, \hat{S}_1, \beta_1, \gamma_1, \eta_1, A_1$ is obtained by assuming $\hat{P} = P, \hat{Q} = Q$ as in Adkins and Paxson (2016), and then finding $\hat{S}_1, \beta_1, \gamma_1, \eta_1, A_1$. An analytical solution is obtained by recognizing that:

$$A_1 = k_{PQ} \hat{P} \hat{Q} / \beta_1 \hat{P}^{\beta_1} \hat{Q}^{\gamma_1} \hat{S}_1^{\eta_1} \tag{12}$$

and

$$\hat{S}_1 = \eta_1 k_{PQ} \hat{P} / \beta_1 k_{SQ} \tag{13}$$

This implies that $\gamma_1 = \beta_1 + \eta_1$ (14)

Eliminating $A_1$ from (8) yields:

$$\beta_1 = k_{PQ} \hat{P} \hat{Q} / (k_{PQ} \hat{P} \hat{Q} + k_{SQ} \hat{S}_1 \hat{Q} - K) \tag{15}$$

So $$\eta_1 = 1 + \beta_1 \left( \frac{K}{k_{PQ} \hat{P} \hat{Q}} - 1 \right) \tag{16}$$

Eliminating $\gamma_1$ and $\eta_1$ from the characteristic root equation (4) yields the quadratic equation:

$$Q(\beta_1) = \beta_1^2 \{a\} + \beta_1 \{b\} - \{c\} = 0 \tag{17}$$

$$a = \left\{ \tfrac{1}{2}\sigma_P^2 - \rho_{PS}\sigma_P\sigma_S + \tfrac{1}{2}\sigma_S^2 + \frac{K^2}{2\hat{P}^2\hat{Q}^2 k_{PQ}^2}[\sigma_Q^2 + 2\rho_{QS}\sigma_Q\sigma_S + \sigma_S^2] + \frac{K}{\hat{P}\hat{Q}k_{PQ}}[\rho_{PQ}\sigma_P\sigma_Q + \rho_{PS}\sigma_P\sigma_S - \rho_{QS}\sigma_Q\sigma_S - \sigma_S^2] \right\}$$

$$b = \{\theta_P - \theta_S - \tfrac{1}{2}\sigma_P^2 - \tfrac{1}{2}\sigma_S^2 + \rho_{PQ}\sigma_P\sigma_Q + \rho_{PS}\sigma_P\sigma_S - \rho_{QS}\sigma_Q\sigma_S + \frac{K}{\hat{P}\hat{Q}k_{PQ}}[\theta_Q + \theta_S + \frac{\sigma_Q^2}{2} + 2\rho_{QS}\sigma_Q\sigma_S + \frac{\sigma_S^2}{2}]\}$$

$$c = -\left\{ r - \theta_Q - \theta_S - \rho_{QS}\sigma_Q\sigma_S \right\}$$

This equation has the simple quadratic solution:

$$\beta_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{18}$$

**Model 5**

**Stochastic Price and Subsidy with a Deterministic Quantity**

We now modify the analysis to consider the impact on the investment decision of a permanent but uncertain government subsidy, denoted by S, but where the output Q sold per unit of time is deterministic.

The PDE is:

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F_2}{\partial P^2} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 F_2}{\partial S^2} + PS\rho_{PS}\sigma_P\sigma_S \frac{\partial^2 F_2}{\partial P \partial S} + \theta_P P \frac{\partial F_2}{\partial P} + \theta_Q Q \frac{\partial F_2}{\partial Q} + \theta_S S \frac{\partial F_2}{\partial S} - rF_2 = 0. \tag{19}$$

where $\theta_X$ denote the risk-neutral drift rates and $r$ the risk-free rate, ($\theta$=r-$\delta$). The solution to (19) is:

$$ROV_2 = F_2 = A_2 P^{\beta_2} Q^{\gamma_2} S^{\eta_2}. \tag{20}$$

where $\beta_2$ , $\gamma_2$ and $\eta_2$ are the power parameters for this option value function (allowing for a deterministic quantity). We expect that all power parameter values are positive. Also, the parameters are linked through the characteristic root equation found by substituting (20) in (19):

$$Q(\beta_2,\gamma_2,\eta_2) = \tfrac{1}{2}\sigma_P^2\beta_2(\beta_2-1) + \tfrac{1}{2}\sigma_S^2\eta_2(\eta_2-1) + \\ +\rho_{PS}\sigma_P\sigma_S\beta_2\eta_2 + \theta_P\beta_2 + \theta_Q\gamma_2 + \theta_S\eta_2 - r = 0 \quad . \tag{21}$$

The value matching relationship becomes:

$$A_2\hat{P}^{\beta_2}\hat{Q}^{\gamma_2}\hat{S}_2^{\eta_2} = k_{PQ}\hat{P}\hat{Q} + k_{SQ}\hat{S}_2\hat{Q} - K \tag{22}$$

Eliminating $\gamma_2$ and $\eta_2$ from the characteristic root equation (21) yields the quadratic equation:

$$Q(\beta_2) = \beta_2^2\{a\} + \beta_2\{b\} - \{c\} = 0 \tag{23}$$

$$a = \left\{\tfrac{1}{2}\sigma_P^2 - \rho_{PS}\sigma_P\sigma_S + \tfrac{1}{2}\sigma_S^2 + \frac{K^2}{2\hat{P}^2\hat{Q}^2k_{PQ}^2}[\sigma_S^2] + \frac{K}{\hat{P}\hat{Q}k_{PQ}}[\rho_{PS}\sigma_P\sigma_S - \sigma_S^2]\right\}$$

$$b = \left\{\theta_P - \theta_S - \tfrac{1}{2}\sigma_P^2 - \tfrac{1}{2}\sigma_S^2 + \rho_{PS}\sigma_P\sigma_S + \frac{K}{\hat{P}\hat{Q}k_{PQ}}[\theta_Q + \theta_S + \frac{\sigma_S^2}{2}]\right\}$$

$$c = -\left\{r - \theta_Q - \theta_S\right\}$$

The solution to this equation is again:

$$\beta_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{24}$$

The difference between (17) and (23) is that the Q volatility has been eliminated, but not the $\theta_Q$.

**Model 6**

**Stochastic Price and Quantity with a Permanent Deterministic Subsidy**

We modify the analysis to consider the impact on the investment decision of a permanent deterministic government subsidy, denoted by S, but where the output Q and market price P are stochastic.

The PDE is:

$$\frac{1}{2}\sigma_P^2 P^2 \frac{\partial^2 F_3}{\partial P^2} + \frac{1}{2}\sigma_Q^2 Q^2 \frac{\partial^2 F_3}{\partial Q^2} + PQ\rho_{PQ}\sigma_P\sigma_Q \frac{\partial^2 F_3}{\partial P \partial Q} + \theta_P P \frac{\partial F_3}{\partial P} + \theta_Q Q \frac{\partial F_3}{\partial Q} + \theta_S S \frac{\partial F_3}{\partial S} - rF_3 = 0. \quad (25)$$

The solution to (25) is:

$$ROV_3 = F_3 = A_3 P^{\beta_3} Q^{\gamma_3} S^{\eta_3}. \quad (26)$$

where $\beta_3$ , $\gamma_3$ and $\eta_3$ are the power parameters for this option value function. The parameters are linked through the characteristic root equation found by substituting (26) in (25):

$$\begin{aligned} Q(\beta_3, \gamma_3, \eta_3) &= \tfrac{1}{2}\sigma_P^2 \beta_3(\beta_3 - 1) + \tfrac{1}{2}\sigma_Q^2 \gamma_3(\gamma_3 - 1) + \\ &\rho_{PQ}\sigma_P\sigma_Q\beta_3\gamma_3 + \theta_P\beta_3 + \theta_Q\gamma_3 + \theta_S\eta_3 - r = 0 \end{aligned} \quad . \quad (27)$$

Eliminating $\gamma_3$ and $\eta_3$ from the characteristic root equation yields the quadratic equation:

$$Q(\beta_3) = \beta_3^2\{a\} + \beta_3\{b\} - \{c\} = 0 \quad (28)$$

$$a = \left\{ \tfrac{1}{2}\sigma_P^2 + \frac{K^2}{2\hat{P}^2\hat{Q}^2 k_{PQ}^2}[\sigma_Q^2] + \frac{K}{\hat{P}\hat{Q}k_{PQ}}[\rho_{PQ}\sigma_P\sigma_Q] \right\}$$

$$b = \{\theta_P - \theta_S - \tfrac{1}{2}\sigma_P^2 + \rho_{PQ}\sigma_P\sigma_Q + \frac{K}{\hat{P}\hat{Q}k_{PQ}}[\theta_Q + \theta_S + \frac{\sigma_Q^2}{2}]\}$$

$$c = -\left\{ r - \theta_Q - \theta_S \right\}$$

The solution to this equation is again: $\beta_3 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ (29)

All of these models can easily be solved in Excel as shown in Figures 5, 6 and 7 below.

Figure 5

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **SUBSIDIES MODEL 4** | |
| 2 | INPUT | Stochastic P & Q & S | | EQ |
| 3 | P | 22.5 | | |
| 4 | Q | 10 | | |
| 5 | S | 30 | per kwh | |
| 6 | R | 325 | B3*B5+B4*B5 | |
| 7 | K | 4000 | | |
| 8 | $\sigma_P$ | **0.20** | | |
| 9 | $\sigma_Q$ | **0.20** | | |
| 10 | σs | **0.20** | | |
| 11 | $\rho_{PQ}$ | 0 | | |
| 12 | $\rho_{PS}$ | 0 | | |
| 13 | $\rho_{SQ}$ | 0 | | |
| 14 | r | 0.08 | | |
| 15 | $\theta_P$ | 0.04 | | |
| 16 | $\theta_Q$ | 0 | | |
| 17 | θs | 0 | | |
| 18 | OUTPUT | **692.08** | B4*(B3+B25) | R* |
| 19 | a1 | 0.0550 | 0.5*(B8^2)+0.5*(B10^2)-B12*B8*B10+((B7^2)/(2*B34))*((B9^2)+2*B13*B9*B10+(B10^2))+B35 | 17 |
| 20 | b1 | 0.0517 | B15-B17-0.5*(B8^2)-0.5*(B10^2)+B11*B8*B9+B12*B8*B10-B13*B9*B10+B36 | 17 |
| 21 | β1 | 0.8244 | (-B20+SQRT((B20^2)-4*B19*(-B14+B16+B17+B13*B9*B10)))/(2*B19) | 18 |
| 22 | η1 | 1.2402 | 1+B21*((B7/(B28*B30*B29))-1) | 16 |
| 23 | γ1 | 2.0646 | B21+B22 | 15 |
| 24 | A1 | 0.0211 | B33/(B21*(B28^B21)*(B29^B23)*(B25^B22)) | 12 |
| 25 | **S^1** | **46.7077** | (B22*B28*B30)/(B21*B32) | 13 |
| 26 | **F1(P,Q,S)** | **2169.7890** | IF(B5<B25, B24*(B3^B21)*(B4^B23)*(B5^B22),B27) | 3 |
| 27 | F1(P,Q,S) | 3757.2233 | (B30*B28*B29)+(B32*B25*B29)-B7 | 8 |
| 28 | P^ | 22.5000 | | |
| 29 | Q^ | 10.0000 | | |
| 30 | P PV rP | 13.7668 | (1-EXP(-(B14-B15)*B38))/(B14-B15) | 5 |
| 31 | Q PV rQ | 9.9763 | (1-EXP(-(B14-B16)*B38))/(B14-B16) | 6 |
| 32 | S PV rS | 9.9763 | (1-EXP(-(B14-B17)*B38))/(B14-B17) | 7 |
| 33 | PQrPQ | 3097.5246 | B28*B29*B30 | |
| 34 | P^2Q^1rPQ^2 | 9594658.5041 | (B28^2)*(B29^2)*(B30^2) | |
| 35 | a2 | -0.0517 | (B7/B33)*(B11*B8*B9+B12*B8*B9-B13*B9*B10-(B10^2)) | 17 |
| 36 | b2 | 0.0517 | (B7/B33)*(B16+B17+0.5*(B9^2)+2*(B13*B9*B10)+0.5*(B10^2)) | 17 |
| 37 | β1 | 0.8244 | B33/(B33+B32*B25*B29-B7) | 15 |
| 38 | T | 20.00000 | | |
| 39 | PDE | 0.0000 | 0.5*(B8^2)*(B3^2)*B43+0.5*(B9^2)*(B4^2)*B44+0.5*(B10^2)*(B5^2)*B45+B15*B3*B40+B16*B4*B41+B17*B5*B42-B14*B26 | 2 |
| 40 | ΔROV1,P | 79.5029 | B21*B24*(B3^(B21-1))*(B4^B23)*(B5^B22) | |
| 41 | ΔROV1,Q | 447.9781 | B23*B24*(B3^B21)*(B4^(B23-1))*(B5^B22) | |
| 42 | ΔROV1,S | 89.6989 | B22*B24*(B3^B21)*(B4^B23)*(B5^(B22-1)) | |
| 43 | ΓROV1,P | -0.6204 | B21*(B21-1)*B24*(B3^(B21-2))*(B4^B23)*(B5^B22) | |
| 44 | ΓROV1,Q | 47.6925 | B23*(B23-1)*B24*(B3^B21)*(B4^(B23-2))*(B5^B22) | |
| 45 | ΓROV1,S | 0.7182 | B22*(B22-1)*B24*(B3^B21)*(B4^B23)*(B5^(B22-2)) | |

Figure 6

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **SUBSIDIES MODEL 5** | |
| 2 | INPUT | **Stochastic P & S** | | |
| 3 | P | 22.50 | | |
| 4 | Q | 10.00 | | |
| 5 | S | 30.00 | per kwh | |
| 6 | R | 325.00 | B3*B5+B4*B5 | |
| 7 | K | 4000.00 | | |
| 8 | $\sigma_P$ | **0.20** | | |
| 9 | $\sigma_Q$ | 0.00 | | |
| 10 | $\sigma S$ | **0.20** | | |
| 11 | $\rho_{PQ}$ | 0.00 | | |
| 12 | $\rho_{PS}$ | 0.00 | | |
| 13 | $\rho_{SQ}$ | 0.00 | | |
| 14 | r | 0.08 | | |
| 15 | $\theta_P$ | 0.04 | | |
| 16 | $\theta_Q$ | 0.00 | | |
| 17 | $\theta s$ | 0.00 | | |
| 18 | OUTPUT | **534.87** | B4*(B3+B25) | R* |
| 19 | a1 | 0.0217 | 0.5*(B8^2)+0.5*(B10^2)-B12*B8*B10+((B7^2)/(2*B34))*((B10^2))+B35 | 23 |
| 20 | b1 | 0.0258 | B15-B17-0.5*(B8^2)-0.5*(B10^2)+B12*B8*B10+B36 | 23 |
| 21 | β2 | 1.4151 | (-B20+SQRT((B20^2)-4*B19*(-B14+B16+B17)))/(2*B19) | 24 |
| 22 | η2 | 1.4123 | 1+B21*((B7/(B28*B30*B29))-1) | 16 |
| 23 | γ2 | 2.8274 | B21+B22 | 14 |
| 24 | A2 | 0.0003 | B33/(B21*(B28^B21)*(B29^B23)*(B25^B22)) | 12 |
| 25 | **S^2** | **30.9869** | (B22*B28*B30)/(B21*B32) | 13 |
| 26 | **F2(P,Q,S)** | **2091.0632** | IF(B5<B25, B24*(B3^B21)*(B4^B23)*(B5^B22),B27) | 20 |
| 27 | F2(P,Q,S) | 2188.8695 | (B30*B28*B29)+(B32*B25*B29)-B7 | 22 |
| 28 | P^ | 22.5000 | | |
| 29 | Q^ | 10.0000 | | |
| 30 | P PV rP | 13.7668 | (1-EXP(-(B14-B15)*20))/(B14-B15) | 5 |
| 31 | Q PV rQ | 9.9763 | (1-EXP(-(B14-B16)*20))/(B14-B16) | 6 |
| 32 | S PV rS | 9.9763 | (1-EXP(-(B14-B17)*20))/(B14-B17) | 7 |
| 33 | PQrPQ | 3097.5246 | B28*B29*B30 | |
| 34 | P^2Q^1rPQ^2 | 9594658.5041 | (B28^2)*(B29^2)*(B30^2) | |
| 35 | a2 | -0.0517 | (B7/B33)*(-(B10^2)) | 23 |
| 36 | b2 | 0.0258 | (B7/B33)*(B16+B17+0.5*(B10^2)) | 23 |
| 37 | β2 | 1.4151 | B33/(B33+B32*B25*B29-B7) | |
| 38 | | | | |
| 39 | PDE | 0.0000 | 0.5*(B8^2)*(B3^2)*B43+0.5*(B10^2)*(B5^2)*B45+B15*B3*B40+B16*B4*B41+B17*B5*B42-B14*B26 | |
| 40 | ΔROV2,P | 131.5163 | B21*B24*(B3^(B21-1))*(B4^B23)*(B5^B22) | |
| 41 | ΔROV2,Q | 591.2329 | B23*B24*(B3^B21)*(B4^(B23-1))*(B5^B22) | |
| 42 | ΔROV2,S | 98.4404 | B22*B24*(B3^B21)*(B4^B23)*(B5^(B22-1)) | |
| 43 | ΓROV2,P | 2.4265 | B21*(B21-1)*B24*(B3^(B21-2))*(B4^B23)*(B5^B22) | |
| 44 | | | | |
| 45 | ΓROV2,S | 1.3529 | B22*(B22-1)*B24*(B3^B21)*(B4^B23)*(B5^(B22-2)) | |

These figures show a different threshold over Models 1-2-3 with some of the same parameter values, because the facility is finite (20 years) rather than perpetual, although the investment opportunity is perpetual. Figure 5 shows a threshold of R*=692, with P,Q and S stochastic. Figure 6 shows a threshold of R*=534 with the same volatility for P and S, but Q is constant. Figure 7 shows R*=673 with a stochastic P and Q (since Q is volatile so is the extra revenue QS, even though S is assumed to be constant). If a government wants to encourage early investment though green certificate allocations, intervening in the certificate trading market to minimize volatility and drift, or an arrangement where the allocation of these certificates is inversely related to Q (which seems fair) would lower the threshold S that justifies immediate investment.

Figure 7

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **SUBSIDIES MODEL 6** | |
| 2 | INPUT | **Stochastic P & Q** | | EQS |
| 3 | P | 22.50 | | |
| 4 | Q | 10.00 | | |
| 5 | S | 30.00 | per kwh | |
| 6 | R | 325.00 | B3*B5+B4*B5 | |
| 7 | K | 4000.00 | | |
| 8 | $\sigma_P$ | **0.20** | | |
| 9 | $\sigma_Q$ | **0.20** | | |
| 10 | σs | 0.00 | | |
| 11 | $\rho_{PQ}$ | 0.00 | | |
| 12 | $\rho_{PS}$ | 0.00 | | |
| 13 | $\rho_{SQ}$ | 0.00 | | |
| 14 | r | 0.08 | | |
| 15 | $\theta_P$ | 0.04 | | |
| 16 | $\theta_Q$ | 0.00 | | |
| 17 | θs | 0.00 | | |
| 18 | OUTPUT | **673.09** | B4*(B3+B25) | R* |
| 19 | a1 | 0.0534 | 0.5*(B8^2)+((B7^2)/(2*B34))*((B9^2))+B35 | 28 |
| 20 | b1 | 0.0458 | B15-B17-0.5*(B8^2)+B11*B8*B9+B36 | 28 |
| 21 | β3 | 0.8682 | (-B20+SQRT((B20^2)-4*B19*(-B14+B16+B17)))/(2*B19) | 29 |
| 22 | η3 | 1.2529 | 1+B21*((B7/(B28*B30*B29))-1) | 16 |
| 23 | γ3 | 2.1211 | B21+B22 | 14 |
| 24 | A3 | 0.0154 | B33/(B21*(B28^B21)*(B29^B23)*(B25^B22)) | 12 |
| 25 | **S^3** | **44.8092** | (B22*B28*B30)/(B21*B32) | 13 |
| 26 | **F3(P,Q,S)** | **2158.1516** | IF(B5<B25, B24*(B3^B21)*(B4^B23)*(B5^B22),B27) | 26 |
| 27 | F3(P,Q,S) | 3567.8196 | (B30*B28*B29)+(B32*B25*B29)-B7 | |
| 28 | P^ | 22.5000 | | |
| 29 | Q^ | 10.0000 | | |
| 30 | P PV rP | 13.7668 | (1-EXP(-(B14-B15)*20))/(B14-B15) | 5 |
| 31 | Q PV rQ | 9.9763 | (1-EXP(-(B14-B16)*20))/(B14-B16) | 6 |
| 32 | S PV rS | 9.9763 | (1-EXP(-(B14-B17)*20))/(B14-B17) | 7 |
| 33 | PQrPQ | 3097.5246 | B28*B29*B30 | |
| 34 | P^2Q^1rPQ^2 | 9594658.5041 | (B28^2)*(B29^2)*(B30^2) | |
| 35 | a2 | 0.0000 | (B7/B33)*(B11*B8*B9+B12*B8*B9) | 28 |
| 36 | b2 | 0.0258 | (B7/B33)*(B16+B17+0.5*(B9^2)) | 28 |
| 37 | β3 | 0.8682 | B33/(B33+B32*B25*B29-B7) | |

Barbosa et al. (2016) re-interpret the role of government as an active player instead of a passive agent, who can undertake the investment but less efficiently and set differential taxes. In this extended model that also captures the multiplier effect of investing, the authors show that the subsidy acts more effectively than a tax reduction in inducing investment, but only up to some maximum level.

### 14.3 Revenue Floors & Ceilings (Real Collar Options)

A real collar option may be a suitable policy device for a government to induce investment by guaranteeing a floor in the face of adverse circumstances, and simultaneously capturing abnormally high returns when the circumstances are sufficiently favourable. Shaoul, Stafford and Stapleton (2012) note that the East Coast Main Line (London to Aberdeen) rail franchise was awarded for a premium paid by the concessionaire, with "clauses that would after four years reimburse the operators for 50% of any shortfall in revenue below 98% of the original forecast and 80% of any shortfall in revenue below 96%, and claw back 50% of any increase in revenue above 102% of the original forecast" (page 13). Implementing a collar results in an earlier exercise due to the guarantee while its cost may be partially reimbursed by penalizing significantly high profits. The analysis of collars adopts a real option formulation because the implied guarantee and penalty are expressible as real options, the sunk cost is partly irretrievable, deferral flexibility is present, and uncertainty prevails. An American perpetuity model produces a straightforward method for engineering a collar because the guarantee level can first be ascertained from knowing the desired threshold prompting exercise, and the penalty level can then be determined from deriving the appropriate ROV (which may, or may not, be paid by the concession investor to the government). American perpetuity and European fixed maturity collars share the characteristic of involving the buying and selling of puts and calls, but the former is also an investment timing model.

There are several authors who have viewed a PPP deal as a set of real options embedded in an active project. Most of these formulations adopt numerical techniques like Monte-Carlo simulation approach sometimes in conjunction with a binomial lattice for obtaining their findings, but some base their conclusions on an analytical real option framework. By evaluating numerically an actual toll road concession involving both a guarantee and penalty, Rose (1998) shows that the government guarantee contributes significant value to the project because returns are conserved at a minimum level. This is replicated using an alternative formulation by Alonso-Conde et al. (2007), who show that these guarantee not only act as incentives but also have the potential of generously transferring significant value to the investor. Cheah and Liu (2006) adopt a similar methodology to reach a similar finding in their investigation of a toll crossing concession. Garvin and Cheah (2004) discuss the advocacy of a real option formulation for capturing the value from deferment and guarantees embedded in PPP deals. The implied value of several interacting flexibilities for a rail concession are investigated by Bowe and Lee (2004), while Huang and Chou (2006) appraise minimum revenue guarantees and abandonment rights for a similar concession using a European-style framework. Blank et al. (2009) investigate the role of a graduated series of guarantees and penalties incurred when operating a toll road concession as a risk transfer device for avoiding bankruptcy that benefits both the investor and lender. Besides these numerical investigations, there are two key analytical studies. Takashima et al. (2010) design a PPP deal involving government debt participation that incorporates a floor on the future maximum loss level where the investor has the right to sell back the project whenever adverse conditions emerge. Using an analytical model, they show the effect of such deals on the investment timing decision. Also, Armada et al. (2012) make an analytical comparison of various subsidy policies and a demand guarantee scheme to reveal their differentiated qualities.

### 14.3.1 Fundamental Model

For a firm in a monopolistic situation confronting a single source of uncertainty due to output price variability, the opportunity to invest in an irretrievable project at cost $K$ depends solely on the price evolution (ignoring operating costs and taxes), which is specified by the geometric Brownian motion process:

$$\mathrm{d}P = \alpha P \mathrm{d}t + \sigma P \mathrm{d}W, \tag{1}$$

where $\alpha$ denotes the expected price risk-neutral drift, $\sigma$ the price volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. Using contingent claims analysis, the option to invest in the project $F(P)$ follows the risk-neutral valuation relationship:

$$\tfrac{1}{2}\sigma^2 P^2 \frac{\partial^2 F}{\partial P^2} + (r-\delta) P \frac{\partial F}{\partial P} - rF = 0, \tag{2}$$

where $r > \alpha$ denotes the risk-free interest rate and $\delta = r - \alpha$ the rate of return shortfall. The generic solution to (2) is:

$$F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2}, \tag{3}$$

where $A_1, A_2$ are to be determined generic constants and $\beta_1, \beta_2$ are, respectively, the positive and negative roots of the fundamental equation, which are given by:

$$\beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}\ . \tag{4}$$

In (3), if $A_2 = 0$ then $F$, a continuously increasing function of $P$, represents an American perpetual call option, Samuelson (1965), while if $A_1 = 0$ then it is a decreasing function and represents a put option, Merton (1973).

In the absence of other forms of optionality and a fixed output volume $Q$, a firm optimally invests when the value matching relationship linking the call option value and the net proceeds $PQ/\delta - K$ is in balance:

$$A_0 P^{\beta_1} = PQ/\delta - K. \quad (5)$$

Following standard methods, the optimal price threshold level triggering investment $\hat{P}_0$ (without collar) is

$$\hat{P}_0 = \frac{\beta_1}{\beta_1 - 1}\frac{\delta}{Q}K \quad (6)$$

and the value function is:

$$F_0(P) = \begin{cases} = \dfrac{K}{\beta_1 - 1}\left(\dfrac{P}{\hat{P}_0}\right)^{\beta_1} & \text{for } P < \hat{P}_0 \\ = \dfrac{PQ}{\delta} - K & \text{for } P \geq \hat{P}_0, \end{cases} \quad (7)$$

with:

$$A_0 = \frac{\hat{P}_0^{1-\beta_1} Q}{\delta \beta_1} = \frac{K \hat{P}_0^{-\beta_1}}{\beta_1 - 1}. \quad (8)$$

### 14.3.2 Investment and Collar Option

A collar option is designed to confine the output price for an active project to a tailored range, by restricting its value to lie between a floor level $P_L$ and a ceiling level $P_H$. Whenever the price falls below the floor, the received output price is assigned the value $P_L$, and whenever it exceeds the ceiling, it is assigned the value $P_H$. By restricting the price to this range, the firm is benefiting by receiving a price that never falls below $P_L$ and is obtaining protection against adverse price movements, whilst at the same time, it is being forced never to receive a price exceeding $P_H$ by sacrificing the upside potential. Protection against downside losses are mitigated in part by sacrificing upside gains. If as part of its subsidy policy, a government offers a firm a price collar in its provision of some output $Q$, the government compensates the firm by a positive amount equalling $(P_L - P)Q$ whenever $P < P_L$, but if the ceiling is breached and $P > P_H$, then the firm reimburses the government by the positive amount $(P - P_H)Q$. It follows that for an active project, the revenue accruing to the firm is given by $\pi_C(P) = \min\{\max\{P_L, P\} P_H\} \times Q$ and its value $V_C$ is described by the risk-neutral valuation relationship:

$$\tfrac{1}{2}\sigma^2 P^2 \frac{\partial^2 V_C}{\partial P^2} + (r-\delta) P \frac{\partial V_C}{\partial P} - rV_C + \pi_C(P) = 0. \tag{9}$$

The relationship (2) and (9) are identical in form except for the revenue function.

The valuation of an active project with a collar is conceived over three mutually exclusive exhaustive regimes, I, II and III, specified on the $P$ line, each with its own distinct valuation function. Regimes I, II and III are defined by $P \le P_L$, $P_L < P \le P_H$ and $P_H \le P$, respectively. Over Regime I, the firm is granted a more attractive fixed price $P_L$ compared with the variable price $P$, but also possesses a call-style option to switch to the more favourable Regime II as soon as $P$ exceeds $P_L$. This switch option increases in value with $P$ and has the generic form $AP^{\beta_1}$, where $A$ denotes a to be determined generic coefficient. Over Regime III, the firm is not only obliged to accept the less attractive fixed price $P_H$ instead of $P$ but also has to sell a put-style option to switch to the less favourable Regime II as soon as $P$ falls below $P_H$. This switch option decreases in value with $P$ and has the generic form $AP^{\beta_2}$. Over Regime II, the firm receives the variable price $P$, possesses a put-style option to switch to the more favourable Regime I as soon as $P$ falls to $P_L$, but sells a call-style option to switch to the less favourable Regime III as soon as $P$ attains $P_H$.

If the subscript $C$ denotes the collar arrangement, then after paying the investment cost, the valuation function for the firm managing the active project is formulated as:

$$V_C\left(P\right)=\begin{cases}\dfrac{P_L Q}{r}+A_{C11}P^{\beta_1} & \text{for } P<P_L\\ \dfrac{PQ}{\delta}+A_{C21}P^{\beta_1}+A_{C22}P^{\beta_2} & \text{for } P_L\le P<P_H\\ \dfrac{P_H Q}{r}+A_{C32}P^{\beta_2} & \text{for } P_H\le P.\end{cases} \tag{10}$$

In (10), for the coefficients the first numerical subscript denotes the regime {1,2,3}, the second subscript denotes a call (if 1) or put (if 2). The coefficients $A_{C11}, A_{C22}$ are expected to be positive because the firm owns the options and a switch is beneficial. In contrast, the $A_{C21}, A_{C32}$ are expected to be negative because the firm is selling the options and is being penalized by the switch. The real collar is composed of a pair of both call and put options. The first pair facilitates switching back and forth between Regimes I and II, which is an advantage for the firm, while the second pair facilitates switching back and forth between Regimes II and III, which is a disadvantage for the firm. The real collar design differs from the typical European collar that only involves buying and selling two distinct options.

A switch between Regimes I and II occurs when $P=P_L$. It is optimal provided the value-matching relationship:

$$\frac{P_L Q}{r}+A_{C12}P^{\beta_2}=\frac{PQ}{\delta}+A_{C21}P^{\beta_1}+A_{C22}P^{\beta_2}, \tag{11}$$

and its smooth-pasting condition expressed as:

$$\beta_2 A_{C12}P^{\beta_2}=\frac{PQ}{\delta}+\beta_1 A_{C21}P^{\beta_1}+\beta_2 A_{C22}P^{\beta_2} \tag{12}$$

both hold when evaluated at $P=P_L$. Similarly, a switch between Regime II and III occurs when $P=P_H$. It is optimal provided the value-matching relationship:

$$\frac{PQ}{\delta}+A_{C21}P^{\beta_1}+A_{C22}P^{\beta_2}=\frac{P_H Q}{r}+A_{C31}P^{\beta_1} \tag{13}$$

and its smooth-pasting condition expressed as:

$$\frac{PQ}{\delta}+\beta_1 A_{C21}P^{\beta_1}+\beta_2 A_{C22}P^{\beta_2}=\beta_1 A_{C31}P^{\beta_1} \tag{14}$$

both hold when evaluated at $P=P_H$. This reveals that:

$$A_{C11}=\left[\frac{P_H Q}{P_H^{\beta_1}}-\frac{P_L Q}{P_L^{\beta_1}}\right]\times\frac{\left(r\beta_2-r-\delta\beta_2\right)}{\left(\beta_1-\beta_2\right)r\delta}>0,\ A_{C21}=\frac{P_H Q\left(r\beta_2-r-\delta\beta_2\right)}{P_H^{\beta_1}\left(\beta_1-\beta_2\right)r\delta}<0,$$
$$A_{C22}=\frac{-P_L Q\left(r\beta_1-r-\delta\beta_1\right)}{P_L^{\beta_2}\left(\beta_1-\beta_2\right)r\delta}>0,\ A_{C32}=\left[\frac{P_H Q}{P_H^{\beta_2}}-\frac{P_L Q}{P_L^{\beta_2}}\right]\times\frac{\left(r\beta_1-r-\delta\beta_1\right)}{\left(\beta_1-\beta_2\right)r\delta}<0. \tag{15}$$

The coefficient $A_{C22}$ for the option to switch from Regime II to I, which depends on only $P_L$ and not on $P_H$, increases in size with $P_L$. This switch option becomes more valuable to the firm as the floor level increases. Similarly, the coefficient $A_{C21}$ for the option to switch from Regime II to III, which depends on only $P_H$ and not on $P_L$, decreases in magnitude with $P_H$. This switch option becomes less valuable to the government as the ceiling level increases. The coefficients $A_{C11}$ and $A_{C32}$ for the switch option from Regime I to II and from Regime III to II, respectively, depend on both $P_L$ and $P_H$.

Figure 8

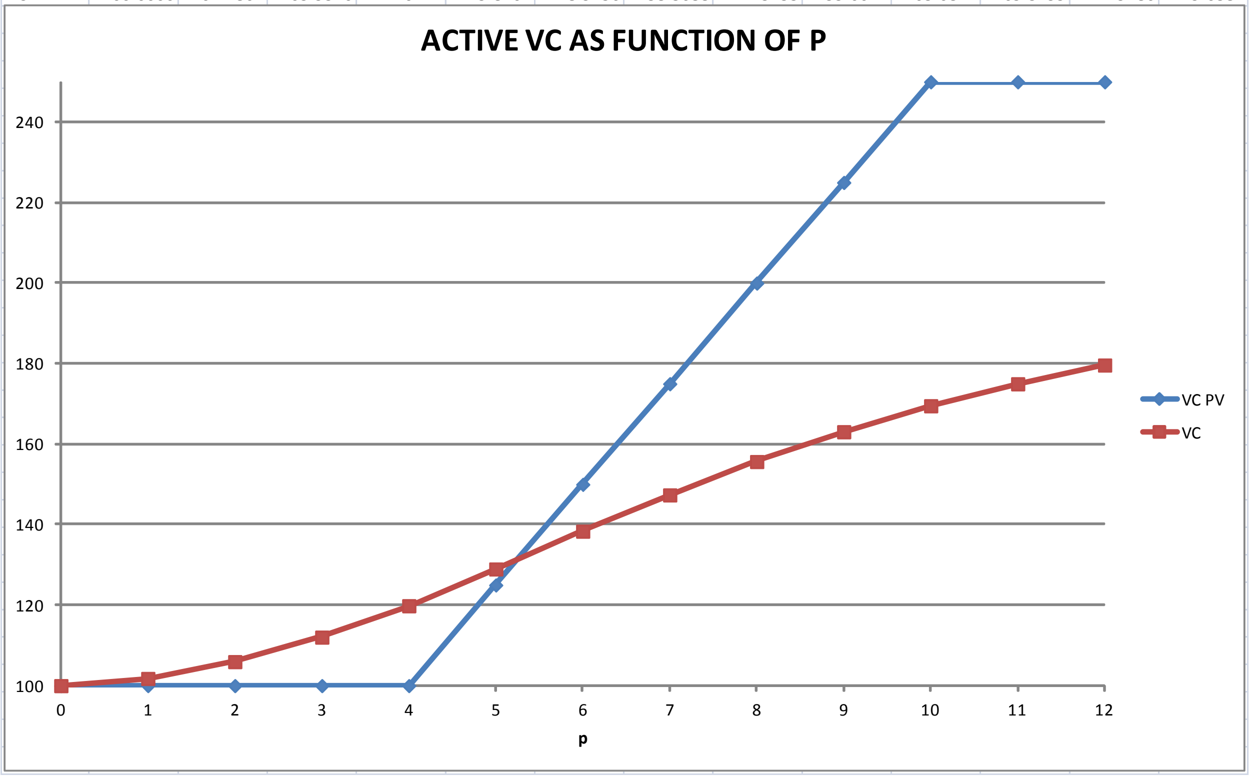
a)

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **ACTIVE PPP WITH COLLAR OPTION** | |
| 2 | **INPUT** | | | EQ |
| 3 | P | 6.00 | | |
| 4 | K | 100.00 | | |
| 5 | σ | 0.25 | | |
| 6 | r | 0.04 | | |
| 7 | δ | 0.04 | | |
| 8 | PL | **4** | | |
| 9 | PH | **10** | | |
| 10 | **OUTPUT** | | | |
| 11 | ROV CALL | 61.8978 | IF(B3<B13,((B4/(B14-1))*(B3/B13)^B14),B12) | 7 |
| 12 | P/δ-K | 50.0000 | MAX(B3/B7-B4,0) | 5 |
| 13 | P^ | 9.4279 | (B14/(B14-1))*B4*B7 | 6 |
| 14 | $\beta_1$ | 1.7369 | 0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2)) | 4 |
| 15 | A0 | 2.7547 | (B4*(B13^-B14))/(B14-1) | 8 |
| 16 | VC | 138.3688 | | 10 |
| 17 | VC PV | 150.0000 | IF(B3<$B$8,$B$8/B6,IF(B3>$B$9,$B$9/B6,B3/B7)) | |
| 18 | β2 | -0.7369 | 0.5-(B6-B7)/(B5^2)-SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2)) | 4 |
| 19 | | | | |
| 20 | | | | |
| 21 | | | | |
| 22 | AC11 | 1.7862 | ($B$9/($B$9^B14)-$B$8/($B$8^B14))*(B26/B28) | 15 |
| 23 | AC21 | -1.8520 | ($B$9/($B$9^B14))*(B26/B28) | 15 |
| 24 | AC22 | 112.2797 | (-$B$8/($B$8^B18))*(B27/B28) | 15 |
| 25 | AC32 | -439.16 | ($B$9/($B$9^B18)-$B$8/($B$8^B18))*(B27/B28) | 15 |
| 26 | [ ] | -0.0400 | (B6*B18-B6-B7*B18) | 15 |
| 27 | ( ) | -0.0400 | (B6*B14-B6-B7*B14) | 15 |
| 28 | { } | 0.0040 | (B14-B18)*B6*B7 | 15 |
| 29 | VC | IF(B3<$B$8,$B$8/B6+B22*(B3^B14),IF(B3>$B$9,$B$9/B6+B25*(B3^B18),B3/B7+B23*(B3^B14)+B24*(B3^B18))) | | |

Figure 8 illustrates that with a floor of 4 and ceiling of 10, and the other parameter values, the option coefficients $A_{C21}$ and $A_{C22}$ are -1.8520 and 112.2797 (15), so the VC equals 138.4 (10) when PL<P<PH, less than the VC PV of 150 which excludes the collar option values.

Figure 9

| P | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| VC PV | 100.0000 | 100.0000 | 100.0000 | 100.0000 | 100.0000 | 125.0000 | 150.0000 | 175.0000 | 200.0000 | 225.0000 | 250.0000 | 250.0000 | 250.0000 |
| VC | 100.0000 | 101.7862 | 105.9540 | 112.0411 | 119.8462 | 128.9756 | 138.3688 | 147.3733 | 155.6677 | 163.0814 | 169.5199 | 174.9786 | 179.6381 |




In Figure 9, past the floor price of 4=PL, the difference between the VC PV and the VC consists of a long position in a put option (should P go below 4) and a short position in a call option (should P rise above 10=PH). If P=6, the net value of the put and call is negative, so the VC PV exceeds the VC. The (VC PV – VC) spread increases as P increases up to 10, the ceiling price.

### 14.3.3 Investment Option

The optimal price threshold $\hat{P}_C$ triggering an investment (with a collar) lies between the floor and ceiling limits, $P_L \le \hat{P}_C \le P_H$. $\hat{P}_C$ attains a minimum of $P_L = rK/Q$ and a maximum of $\hat{P}_0$ for $P_L = 0$, so the introduction of a price floor always produces at least an hastening of the investment exercise and never its postponement. The ceiling limit holds because of the absence of any effective economic benefit from exercising at a price exceeding the ceiling. The following analysis treats the threshold $\hat{P}_C$ as lying between the lower and upper limits. When $P_L \le \hat{P}_C \le P_H$, the optimal solution is obtained from equating the investment option value with the active project net value at the threshold $P = \hat{P}_C$. The optimal solution is determined from both the value-matching relationship:

$$A_{C0}P^{\beta_1} = \frac{PQ}{\delta} + A_{C21}P^{\beta_1} + A_{C22}P^{\beta_2} - K \tag{16}$$

and its smooth-pasting condition expressed as:

$$\beta_1 A_{C0}P^{\beta_1} = \frac{PQ}{\delta} + \beta_1 A_{C21}P^{\beta_1} + \beta_2 A_{C22}P^{\beta_2} \tag{17}$$

when evaluated for $P = \hat{P}_C$. This reveals that:

$$\frac{\hat{P}_C Q}{\delta} = \frac{\beta_1}{\beta_1 - 1}K - \frac{\beta_1 - \beta_2}{\beta_1 - 1}A_{C22}\hat{P}_C^{\beta_2}, \tag{18}$$

$$\begin{aligned} A_{C0} &= \frac{K\hat{P}_C^{-\beta_1}}{\beta_1 - 1} - \left(\frac{1-\beta_2}{\beta_1 - 1}\right)A_{C22}\hat{P}_C^{\beta_2-\beta_1} + A_{C21} \\ &= \frac{1}{\beta_1 - \beta_2}\left[(1-\beta_2)\frac{\hat{P}_C Q}{\delta} + \beta_2 K\right]\hat{P}_C^{-\beta_1} + A_{C21}. \end{aligned} \tag{19}$$

Since a closed form solution for $\hat{P}_C$ does not exist, equation (18) is solved numerically for $\hat{P}_C$ and then equation (19) for $A_{C0}$. The investment value for the project is:

$$F_{C0}(P) = \begin{cases} A_{C0}P^{\beta_1} & \text{for } P < \hat{P}_C \\ \dfrac{PQ}{\delta} - K + A_{C21}P^{\beta_1} + A_{C22}P^{\beta_2} & \text{for } \hat{P}_C \le P < P_H, \end{cases} \tag{20}$$

where $P_L \le \hat{P}_C \le P_H$.

From (18), the threshold $\hat{P}_C$ depends only on the floor $P_L$ through $A_{C22}$, but not on the ceiling $P_H$. Adjusting the ceiling of the collar has no material impact on the threshold, so the timing decision is affected by the losses foregone by having a floor but not by the gains sacrificed by having a ceiling. Since $A_{C22}$ is non-negative, the with-collar threshold $\hat{P}_C$ is always no greater than the without-collar threshold $\hat{P}_0$ (6), and an increase in the floor produces an earlier exercise due to the reduced threshold level.

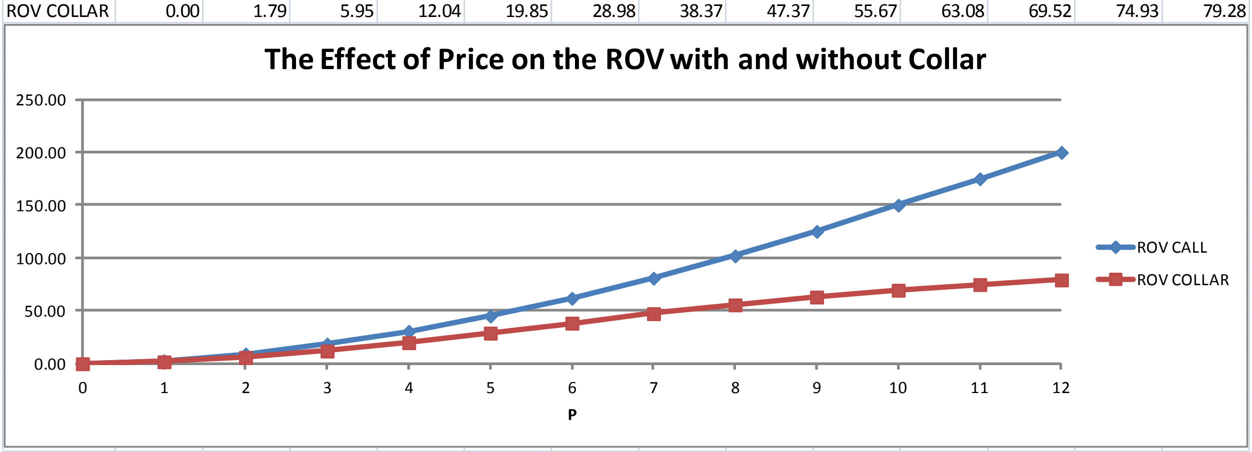
Figure 10 shows that with a floor of 4 and ceiling of 10, and the other parameter values, the option coefficients $A_{C21}$ and $A_{C22}$ are -1.8520 and 112.2797 (15), so the FC is 38.4 (20) when PL<P<PH, less than the ROV without collar 61.9 (7).

Figure 10

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | | **INVESTMENT OPPORTUNITY FOR A PPP WITH A COLLAR OPTION** | |
| 2 | **INPUT** | | | EQ |
| 3 | P | 6.00 | | |
| 4 | K | 100.00 | | |
| 5 | σ | 0.25 | | |
| 6 | r | 0.04 | | |
| 7 | δ | 0.04 | | |
| 8 | PL | **4** | | |
| 9 | PH | **10** | | |
| 10 | **OUTPUT** | | | |
| 11 | ROV CALL | 61.8978 | IF(B3<B13,((B4/(B14-1))*(B3/B13)^B14),B12) | 7 |
| 12 | P/δ-K | 50.0000 | MAX(B3/B7-B4,0) | 5 |
| 13 | P^ | 9.4279 | (B14/(B14-1))*B4*B7 | 6 |
| 14 | $\beta_1$ | 1.7369 | 0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2)) | 4 |
| 15 | A0 | 2.7547 | (B4*(B13^-B14))/(B14-1) | 8 |
| 16 | | | | 10 |
| 17 | ROV COLLAR | 38.3688 | IF(B3<B20,B21*(B3^B14),B3/B7-B4+B23*(B3^B14)+B24*(B3^B18)) | 20 |
| 18 | β2 | -0.7369 | 0.5-(B6-B7)/(B5^2)-SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2)) | 4 |
| 19 | FIND P^ | 0.0000 | B20/B7-(B14/(B14-1))*B4+((B14-B18)/(B14-1))*B24*(B20^B18) | 18 |
| 20 | P^ | 4.0000 | | |
| 21 | AC0 | 1.7862 | (1/(B14-B18))*((1-B18)*(B20/B7)+B18*B4)*(B20^-B14)+B23 | 19 |
| 22 | AC11 | | ($B$9/($B$9^B14)-$B$8/($B$8^B14))*(B26/B28) | 15 |
| 23 | AC21 | -1.8520 | ($B$9/($B$9^B14))*(B26/B28) | 15 |
| 24 | AC22 | 112.2797 | (-$B$8/($B$8^B18))*(B27/B28) | 15 |
| 25 | AC32 | | ($B$9/($B$9^B18)-$B$8/($B$8^B18))*(B27/B28) | 15 |
| 26 | [    ] | -0.0400 | (B6*B18-B6-B7*B18) | 15 |
| 27 | (    ) | -0.0400 | (B6*B14-B6-B7*B14) | 15 |
| 28 | {    } | 0.0040 | (B14-B18)*B6*B7 | 15 |

Figure 11

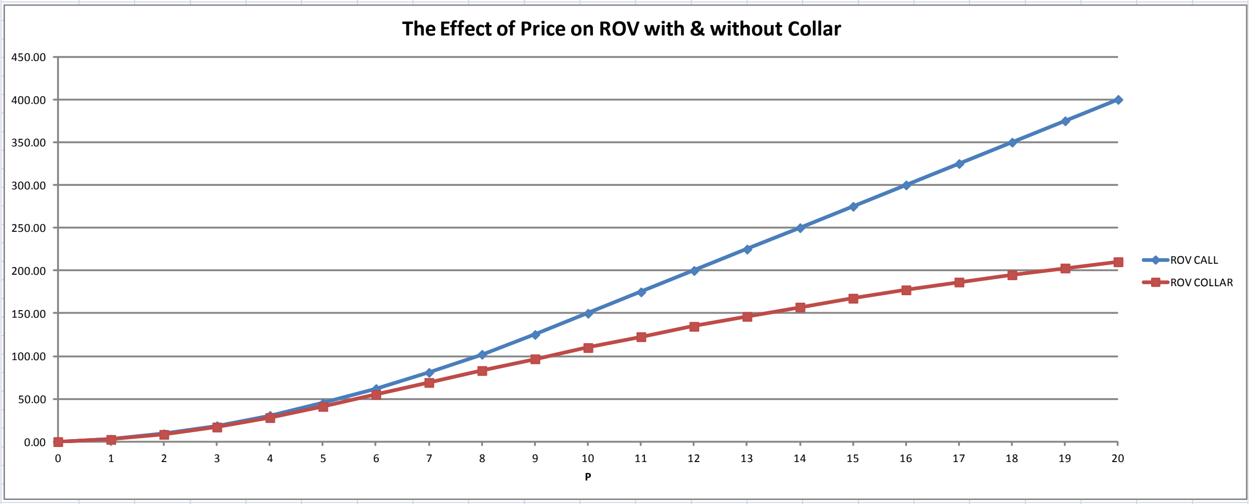
| P | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ROV CALL | 0.00 | 2.75 | 9.18 | 18.57 | 30.61 | 45.10 | 61.90 | 80.90 | 102.02 | 125.18 | 150.00 | 175.00 | 200.00 |
| ROV COLLAR | 0.00 | 1.79 | 5.95 | 12.04 | 19.85 | 28.98 | 38.37 | 47.37 | 55.67 | 63.08 | 69.52 | 74.93 | 79.28 |



In Figure 11, the ROV Collar (PL=4, PH=10) always has a lower value than a standard ROV without a collar, since there is no upper limit to the investment profit, and the investment opportunity is an option, not yet a commitment.

Figure 12

| P | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ROV CALL | 0.00 | 2.75 | 9.18 | 18.57 | 30.61 | 45.10 | 61.90 | 80.90 | 102.02 | 125.18 | 150.00 | 175.00 | 200.00 | 225.00 | 250.00 | 275.00 | 300.00 | 325.00 | 350.00 | 375.00 | 400.00 |
| ROV COLLAR | 0.00 | 2.53 | 8.42 | 17.03 | 28.08 | 41.10 | 55.01 | 69.13 | 83.10 | 96.74 | 109.94 | 122.63 | 134.76 | 146.32 | 157.28 | 167.64 | 177.38 | 186.51 | 195.03 | 202.94 | 210.23 |




PL 4 PH 20

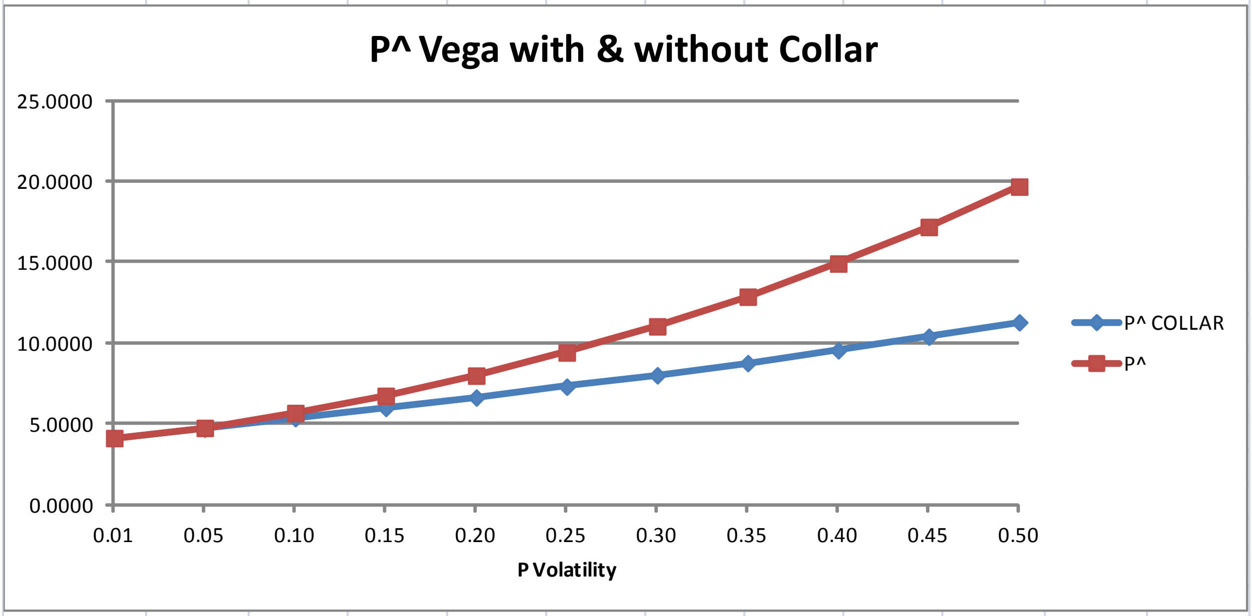
In Figure 12, the ROV Collar with a higher price ceiling, in this case PH=20, is more valuable than with the previous ceiling of PH=10, and the spread between the ROV with and without collar increases as P approaches PH.

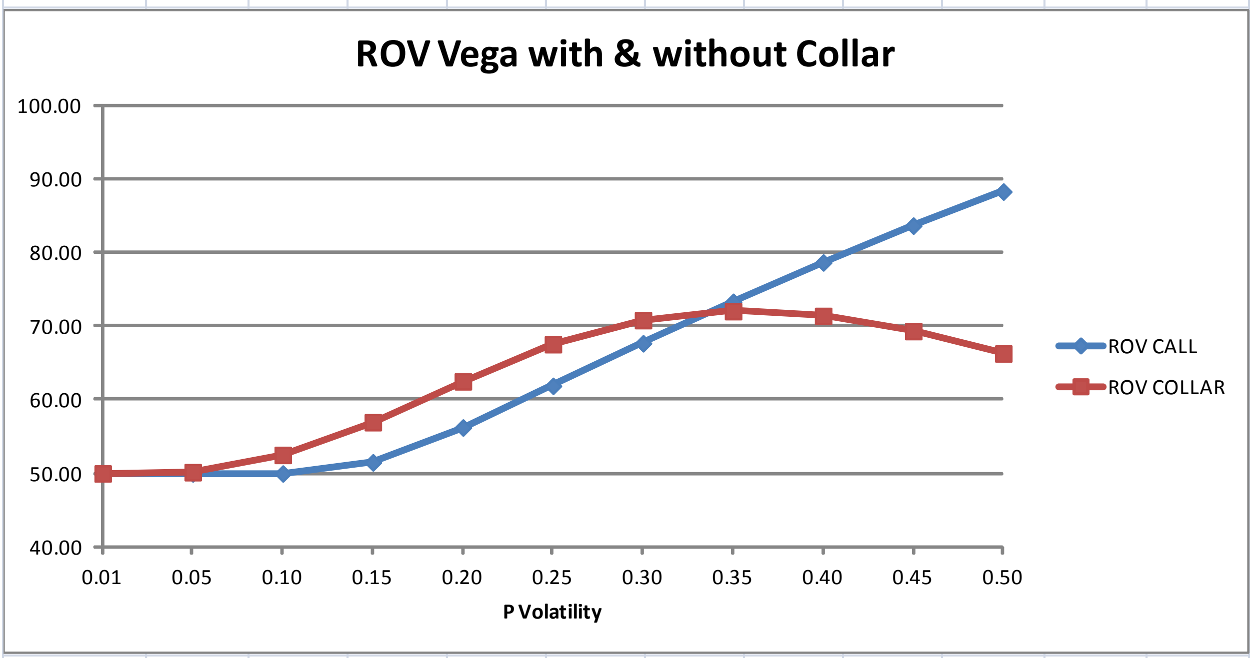
One of the most interesting aspects of comparing simple real investment options with real investment options with a collar is the effect of increasing P volatility on the price threshold that justifies immediate investment, and also on the ROV (the so-called "vega"). Naturally the price threshold increases with the increased of expected price volatility shown in Figure 13, so a government seeking early investment might consider imposing a collar in a volatile price environment. To the extent that this price is correlated with traded futures or securities, so the prospective concessionaire might seek to hedge this volatility, a collar seems less relevant, or in a low price volatility environment redundant (as regards the price threshold). Note, this illustration assumes a very high price ceiling. The ROV without a collar increases almost linearly with increases in the price volatility, but the ROV with a collar has a different pattern as in Figure 13. From a low volatility environment, the ROV + Collar increases, but eventually at high expected volatilities the vega almost becomes negative, due to the increase in the value of the written call option. Whether this holds if the price volatility can be hedged is an interesting question.

Note, this illustration assumes a very high price ceiling; the ROV+ collar vega is different for different floors and ceilings.

Figure 13

| σ | 0.01 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ROV CALL | 50.00 | 50.00 | 50.00 | 51.51 | 56.25 | 61.90 | 67.69 | 73.32 | 78.66 | 83.65 | 88.28 |
| ROV COLLAR | 50.00 | 50.18 | 52.52 | 56.97 | 62.51 | 67.55 | 70.82 | 72.03 | 71.40 | 69.34 | 66.30 |
| P^ COLLAR | 4.1439 | 4.7168 | 5.3681 | 6.0008 | 6.6458 | 7.3178 | 8.0254 | 8.7739 | 9.5670 | 10.4074 | 11.2970 |
| P^ | 4.1439 | 4.7724 | 5.6861 | 6.7571 | 8.0000 | 9.4279 | 11.0523 | 12.8831 | 14.9282 | 17.1945 | 19.6873 |




| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 2 | | PL | **3** | | | | | | PH | **500** |

EXERCISE 14.1

Sonja believes she can build a solar plant for K=$4000 that will produce Q=10 KWh per year, that can be sold for P=$10 per KWh, P*Q=R. $ROV = B_1 R^{\beta_1}$, where $\beta_1$=2. For a subsidy $\tau$, the threshold $\hat{R}$ that justifies immediate investment is: $\hat{R} = \frac{\beta_1}{\beta_1 - 1} K \frac{(r - \delta_R)}{(1+\tau)}, B_1 = \frac{(1+\tau)\hat{R}^{1-\beta_1}}{\beta_1 (r - \delta_R)}$. If r=.07, electricity $\delta$ =.04, a proportional subsidy $\tau$ =1, should Sonja build now, or try to sell this opportunity for $2500?

EXERCISE 14.2

Carlos Azevedo owns the same type of solar plant that Sonja hopes to build, with a constant Q=1 KWh per year, the electricity price =€ 2, but the generous Portuguese government has guaranteed a revenue of € 4 per annum but required a ceiling of € 10. If r=.04, electricity $\delta$ =.04, σ=25%, should Carlos try to sell this plant for €100, if $A_{C11}$=1.7862 ? $V_C(P) = \frac{P_L Q}{r} + A_{C11} P^{\beta_1}$ for $P < P_L$

$$\beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} .$$

EXERCISE 14.3

Susanne Das owns the same type of solar plant that Sonja hopes to build, with a constant Q=1 KWh per year, the electricity price =€ 12, but a mean Spanish government requires Susanne to donate all revenue over € 10 per annum to the local resting home for old bulls. If r=.04, electricity $\delta$ =.04, σ=25%, should Susanne try to sell this plant for €200, if $A_{C32}$=-439.16 ?

$$V_C(P) = \frac{P_H Q}{r} + A_{C32} P^{\beta_2} \quad \text{for } P_H < P . \quad \beta_1, \beta_2 = \left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right) \pm \sqrt{\left(\tfrac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$$

PROBLEM 14.4

Sonja believes she can build a solar plant for K=4000 that will produce Q=10 KWh per year, that can be sold for P=22.25 per KWh, P*Q=R. $ROV = B_1 R^{\beta_1}$, where $\beta_1$ is the solution to a simple quadratic equation. For a proportional subsidy $\tau$, the threshold $\hat{R}$ that justifies immediate investment is:

$$\hat{R} = \frac{\beta_1}{\beta_1 - 1} K \frac{(r-\delta_R)}{(1+\tau)}, B_1 = \frac{(1+\tau)\hat{R}^{1-\beta_1}}{\beta_1(r-\delta_R)}, \beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{(\frac{r-\delta}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$$

If r=.08, electricity $\delta$ =.04, R volatility=.2, subsidy $\tau$ =.10, what R would justify immediate investment, and what is the value of this investment opportunity?

PROBLEM 14.5

Carlos Azevedo owns the same type of solar plant that Sonja hopes to build, with a constant Q=1 KWh per year, the electricity price =€ 4, but the generous Portuguese government has guaranteed a revenue of € 6 per annum but required a ceiling of € 20. If r=.04, electricity $\delta$ =.04, σ=25%, should Carlos try to sell this plant for €150, if $A_{C11} = \left[ \frac{P_H Q}{P_H^{\beta_1}} - \frac{P_L Q}{P_L^{\beta_1}} \right] \times \frac{(r\beta_2 - r - \delta\beta_2)}{(\beta_1 - \beta_2) r\delta} > 0$ and

$V_C(P) = \frac{P_L Q}{r} + A_{C11} P^{\beta_1}$ for $P < P_L$ ?

PROBLEM 14.6

Susanne Das owns the same type of solar plant that Sonja hopes to build, with a constant Q=1 KWh per year, the electricity price =€ 14, but a mean Spanish government requires Susanne to donate all revenue over € 12 per annum to the local resting home for old bulls in return for minimum guaranteed price of € 4. If r=.04, electricity $\delta$ =.04, σ=25%, should Susanne try to sell this plant for € 200, if $A_{C32} = \left[ \frac{P_H Q}{P_H^{\beta_2}} - \frac{P_L Q}{P_L^{\beta_2}} \right] \times \frac{(r\beta_1 - r - \delta\beta_1)}{(\beta_1 - \beta_2) r\delta} < 0$. and $V_C(P) = \frac{P_H Q}{r} + A_{C32} P^{\beta_2}$ for $P_H < P$ ?

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